PHYS 144 – Error Propagation Examples

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1. Derive an expression for uncertainty in acceleration due to gravity from a position-time graph.

The equation for a second-order polynomial is:

$$f(x) = a_2 x^2 + a_1 x + a_0. \tag{1}$$

The equation for position as a function of time is:

$$y(t) = -\frac{1}{2} g t^2 + v_0 t + y_0. \tag{2}$$

Comparing (1) and (2), it is clear that $a_2 = -(1/2)g$. Rearranging this, g is given by:

$$g = -2a_2.$$

`np.polyfit()` gives the uncertainty of a_2, δa_2, which will propagate into uncertainty in g. Using the general error propagation formula on page 33 of the lab manual, uncertainty in g will be given by:

$$\delta g = \sqrt{\left(\frac{\partial g}{\partial a_2}\right)^2 (\delta a_2)^2}. \tag{3}$$

The part in the first set of brackets is a partial derivative. To compute partial derivatives, treat every quantity that is not being differentiated with respect to as a constant. Because the expression for g only contains one quantity (a_2), the partial derivative will be the same as the total derivative. Computing the derivative:

$$\left(\frac{\partial g}{\partial a_2}\right) = -2. \tag{4}$$

Plugging (4) into (3) and simplifying:

$$\boxed{\delta g = 2\delta a_2}.$$

The negative sign in (4) goes away because the partial derivative is squared in (3). The result is that the uncertainty in g is twice the uncertainty in a_2, because g is twice as big as a_2. This is how error propagation works for simple multiplication by a constant, but it gets more complicated when you have equations with multiple quantities. See the more difficult examples below.

2. Derive an expression for the error in instantaneous velocity. Is error constant for all values of v?

The equation for instantaneous velocity is:

$$v = \frac{x_{n+1} - x_{n-1}}{t_{n+1} - t_{n-1}}.$$

Assume uncertainty in time is negligible. Then, only the errors in x_{n+1} and x_{n-1} will propagate into the error of v. From the general error propagation formula, uncertainty in v is given by:

$$\delta v = \sqrt{\left(\frac{\partial v}{\partial x_{n+1}}\right)^2 \delta x_{n+1}^2 + \left(\frac{\partial v}{\partial x_{n-1}}\right)^2 \delta x_{n-1}^2}. \tag{1}$$

Computing the derivatives:

$$\left(\frac{\partial v}{\partial x_{n\pm1}}\right) = \frac{\pm 1}{t_{n+1} - t_{n-1}}, \tag{2}$$

Plugging (2) into (1) and simplifying:

$$\boxed{\delta v = \left(\frac{1}{t_{n+1} - t_{n-1}}\right)\sqrt{\delta x_{n+1}^2 + \delta x_{n-1}^2}}. \tag{3}$$

If more than one meter stick is used to measure x_n, some x_n values will have larger uncertainty than others (δx_{n+1} and δx_{n-1} will not necessarily be equal), meaning the error will not be constant for all values of v.

3. Derive an expression for uncertainty in total mechanical energy.

The equation for total mechanical energy is given by:

$$E = K + U,$$

where K is kinetic energy and U is gravitational potential energy.
From basic error propagation, the uncertainty in E is given by:

$$\boxed{\delta E = \sqrt{(\delta K)^2 + (\delta U)^2}}, \tag{1}$$

where δK is the uncertainty in kinetic energy, and δU is the uncertainty in potential energy, both of which must be calculated. Begin with δK.

Kinetic energy is given by:

$$K = \frac{1}{2}mv^2,$$

where there is uncertainty in both m and v. Using the general error propagation formula, δK can be calculated with:

$$\delta K = \sqrt{\left(\frac{\partial K}{\partial m}\right)^2 (\delta m)^2 + \left(\frac{\partial K}{\partial v}\right)^2 (\delta v)^2}. \tag{2}$$

Computing the derivatives:

$$\left(\frac{\partial K}{\partial m}\right) = \frac{1}{2}v^2,$$

$$\left(\frac{\partial K}{\partial v}\right) = mv,$$

and then substituting these into (2) and simplifying (by multiplying the first term under the square root by m/m, and the second by $2v/2v$), δK becomes:

$$\delta K = \sqrt{\left(\frac{mv^2}{2m}\right)^2 (\delta m)^2 + \left(\frac{2mv^2}{2v}\right)^2 (\delta v)^2},$$

$$\boxed{\delta K = K\sqrt{\left(\frac{\delta m}{m}\right)^2 + \left(\frac{2\delta v}{v}\right)^2}}, \tag{3}$$

where δm is the uncertainty in the mass, and δv is the uncertainty in the magnitude of velocity, which must be calculated.

The magnitude of the velocity is given by:

$$v = \sqrt{v_x^2 + v_y^2},$$

where there is uncertainty in both v_x and v_y. Using the general error propagation formula, δv can be calculated with:

$$\delta v = \sqrt{\left(\frac{\partial v}{\partial v_x}\right)^2 (\delta v_x)^2 + \left(\frac{\partial v}{\partial v_y}\right)^2 (\delta v_y)^2}. \tag{4}$$

Computing the derivatives:

$$\left(\frac{\partial v}{\partial v_{x,\ y}}\right) = \frac{1}{2}(v_x^2 + v_y^2)^{-1/2} \cdot 2v_{x,\ y} = \frac{v_{x,\ y}}{v},$$

and then substituting these into (4) and simplifying, δv becomes:

$$\boxed{\delta v = \frac{1}{v}\sqrt{(v_x \delta v_x)^2 + (v_y \delta v_y)^2}}, \tag{5}$$

where δv_x and δv_y are the uncertainties in the x and y components of the velocity. These can be calculated using Equation (3) from the previous example:

$$\boxed{\delta v_x = \left(\frac{1}{t_{n+1} - t_{n-1}}\right)\sqrt{\delta x_{n+1}^2 + \delta x_{n-1}^2}}, \tag{6}$$

and the same for y.

Equation (6) (and the equivalent equation for y) can now be substituted into Equation (5), which can be substituted into Equation (3), which can be substituted into Equation (1). This completes the steps for the error propagation for kinetic energy. The value of δK will be different for each data point, as it depends on v, which depends on x and y.

The next step is the error propagation for potential energy. Potential energy is given by:

$$U = mgy,$$

where there is uncertainty in all variables. Using the general error propagation formula, δU can be calculated with:

$$\delta U = \sqrt{\left(\frac{\partial U}{\partial m}\right)^2 (\delta m)^2 + \left(\frac{\partial U}{\partial g}\right)^2 (\delta g)^2 + \left(\frac{\partial U}{\partial y}\right)^2 (\delta y)^2}. \tag{7}$$

Computing the derivatives:

$$\left(\frac{\partial U}{\partial m}\right) = gy,$$

$$\left(\frac{\partial U}{\partial g}\right) = my,$$

$$\left(\frac{\partial U}{\partial y}\right) = mg,$$

and then substituting these into (7) and simplifying (by multiplying each term under the square root by mgy/mgy), δU becomes:

$$\delta U = \sqrt{\left(\frac{mgy}{m}\right)^2 (\delta m)^2 + \left(\frac{mgy}{g}\right)^2 (\delta g)^2 + \left(\frac{mgy}{y}\right)^2 (\delta y)^2}$$

$$\boxed{\delta U = U\sqrt{\left(\frac{\delta m}{m}\right)^2 + \left(\frac{\delta g}{g}\right)^2 + \left(\frac{\delta y}{y}\right)^2}}, \tag{8}$$

where δg is the uncertainty in acceleration due to gravity. Equation (8) can now be substituted into Equation (1). This completes the steps for the error propagation for potential energy. The value of δU will be different for each data point, as it depends on y.

This completes the steps for the error propagation for total mechanical energy.